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Time Required to Unify All Particles in the Scheme of Equiprobable Allocation into a Sequence of Cell Layers

A. M. Zubkov[1*] and O. K. Shibanov[2]**

[1] *Steklov Mathematical Institute, Russian Academy of Sciences*

[2] *Moscow State University*

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Abstract—We study the scheme of equiprobable allocations of particles into a sequence of cell layers, where the particles put into the same cell are considered as a single particle. We present conditions under which there exist, with positive probability, nonunified particles at each of the layers. For the case in which the number of cells at each of the layers is equal to the number of original particles, we prove the limit theorem for the time instant at which all the particles are unified into a single particle.

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1. INTRODUCTION

We consider the process of allocation of particles into cell layers of the following form. At the first stage, n_0 original particles are independently and equiprobably allocated into n_1 cells of the first layer. The particles put into the same cell of the first layer are unified into a new particle; in this case, a random number ψ_1 of unified particles is obtained at the first layer (which is equal to the number of cells of the first layer occupied by the original particles). In the general case, at the $(k+1)$th stage, the ψ_k unified particles located in n_k cells of the kth layer are independently (from each other and from the prehistory) and equiprobably allocated into n_{k+1} cells of the $(k+1)$th layer; the particles put into the same cell of the $(k+1)$th layer are unified, and ψ_{k+1} is the number of nonempty cells of the $(k+1)$th layer. Under these assumptions, the sequence $\psi_0, \psi_1, \dots$ forms a Markov chain with nonincreasing trajectories.

Multistage schemes of allocation of particles were introduced in [1]. In [2] and [3], the Poisson limit theorems were proved for the two-stage scheme of allocation of particles into cells; in [4], the central limit theorem was formulated for the two-stage allocation scheme. In [5], for the two-stage allocation scheme, the generating function of the number of the second layer cells, which contain exactly r original particles, was obtained for $n_0 = n_1 = n_2$.

Theorem 1. *If* $n_* = \min_{k\ge 0} n_k \ge 2$, *then*

$$\mathsf{P}\Big\{\lim_{k\to\infty} \psi_k > 1\Big\} > 0 \qquad \Longleftrightarrow \qquad \sum_{k=1}^{\infty} \frac{1}{n_k} < \infty.$$

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*E-mail: zubkov@mi.ras.ru
**E-mail: oshi@yandex.ru

In what follows, we consider the case in which all layers are of the same size, i.e., $n_0 = n_1 = n_2 = \dots = n$. Then, by Theorem 1, with probability 1, all n original particles are unified in finitely many steps, i.e., the first time instant τ_n at which all the particles are unified into a single particle has its own distribution. We note that, for $n_0 = n_1 = \dots = n$, another natural interpretation of this variable is possible. If $S = \{1, \dots, n\}$ and $f_1, f_2, \dots$ are independent equiprobable maps $S \to S$, then

$$\tau_n = \min\{T : |f_T(f_{T-1}(\dots f_1(S) \dots))| = 1\}$$

is the minimal number of iterations of random maps S into itself for which the image of S is a one-element set.

Theorem 2. *As $n \to \infty$, the distributions of random variables $\zeta_n = \tau_n/n$ converge to the distribution of the sum $\xi = \sum_{j=1}^{\infty} \xi_j$, where the random variables $\xi_1, \xi_2, \dots$ are independent and*

$$\mathsf{P}\{\xi_j \le x\} = 1 - e^{-xj(j+1)/2}, \quad x \ge 0, \qquad j = 1, 2, \dots .$$

Remark. Since $\mathsf{E}\xi_j = 2/(j(j+1))$, the random variable ξ has a finite mathematical expectation:

$$\mathsf{E}\xi = \sum_{j=1}^{\infty} \frac{2}{j(j+1)} = 2.$$

In Sec. 2 of the present paper, we prove Theorem 1, and in Sec. 3, Theorem 2.

2. CONDITIONS FOR THE COMPLETE UNIFICATION OF PARTICLES

We assume that the conditions of Theorem 1 are satisfied, and we set $t_* = \min\{k \ge 1 : n_k = n_*\}$.

The sequence ψ_k forms a Markov chain (inhomogeneous in time) with nonincreasing trajectories. It is easy to verify that, for any $k, m > 0$,

$$\mathsf{P}\{\psi_{k+1} = m \mid \psi_k = m\} = \frac{n_{k+1}^{[m]}}{n_{k+1}^{m}} \qquad \text{if} \quad n_{k+1} \ge m, \tag{1}$$

where $n^{[m]} = n(n-1)\cdots(n-m+1)$; in addition,

$$\mathsf{P}\{\psi_k = n_* \text{ for all } k \ge t_*\} = \mathsf{P}\{\psi_{t_*} = n_*\}\, \mathsf{P}\{\psi_k = n_* \text{ for all } k > t_* \mid \psi_{t_*} = n_*\}, \tag{2}$$

and the first multiplier in the right-hand side is positive.

Lemma 1. *For $n_* \ge 2$ and any $t \ge 1$ and $h \ge 2$ such that $n_k \ge h$ for all $k \ge t$,*

$$\sum_{k=1}^{\infty} \frac{1}{n_k} < \infty \qquad \Longleftrightarrow \qquad \mathsf{P}\{\psi_k = h \textit{ for all } k > t \mid \psi_t = h\} > 0.$$

Proof. Since the sequence $\{\psi_n\}$ is a Markov chain, we have

$$\mathsf{P}\{\psi_k = h \text{ for all } k \ge t \mid \psi_t = h\} = \prod_{k=t}^{\infty} \mathsf{P}\{\psi_{k+1} = h \mid \psi_k = h\} = \prod_{k=t}^{\infty} \frac{n_{k+1}^{[h]}}{n_{k+1}^{h}} .$$

The fact that the infinite product converges to a nonzero value is equivalent to the convergence of the series of logarithms of the multipliers:

$$\sum_{k=t}^{\infty} \ln\biggl(\frac{n_k(n_k-1)\cdots(n_k-h+1)}{n_k^h}\biggr). \tag{3}$$

Since

$$\frac{(n_k-h+1)^h}{n_k^h} \le \frac{n_k(n_k-1)\cdots(n_k-h+1)}{n_k^h} \le \frac{n_k-1}{n_k}$$

for $h \geq 2$, we have

$$h \sum_{k=t}^{\infty} \ln\left(\frac{n_k - h + 1}{n_k}\right) \leq \sum_{k=t}^{\infty} \ln\left(\frac{n_k(n_k-1)\cdots(n_k-h+1)}{n_k^h}\right) \leq \sum_{k=t}^{\infty} \ln\left(\frac{n_k - 1}{n_k}\right). \tag{4}$$

For $n_* \geq 2$, the series in the left- and right-hand sides of (4) converge or diverge simultaneously with the series $\sum_{k=1}^{\infty} 1/n_k$, and hence the convergence of this series is equivalent to the fact that the condition $\mathsf{P}\{\psi_k = h \text{ for all } k \geq t \mid \psi_t = h\} > 0$ is satisfied for $h \geq 2$. The proof of Lemma 1 is complete. □

Thus, if $\sum_{k=1}^{\infty} 1/n_k < \infty$, then

$$\mathsf{P}\Big\{\lim_{k\to\infty} \psi_k > 1\Big\} > 0 \geq \mathsf{P}\Big\{\lim_{k\to\infty} \psi_k = n_*\Big\} > 0.$$

Now we assume that $\sum_{k=1}^{\infty} 1/n_k = \infty$. Since the sequence $\{\psi_k\}$ does not increase, we have

$$\mathsf{P}\Big\{\lim_{k\to\infty} \psi_k > 1\Big\} = \sum_{h=2}^{n_*} \mathsf{P}\Big\{\lim_{k\to\infty} \psi_k = h\Big\}$$

and, for any $h = 2, \dots, n_*$,

$$\mathsf{P}\Big\{\lim_{k\to\infty} \psi_k = h\Big\} \leq \sum_{t=t_*}^{\infty} \mathsf{P}\{\psi_k = h \text{ for all } k \geq t\} \leq \sum_{t=t_*}^{\infty} \mathsf{P}\{\psi_k = h \text{ for all } k > t \mid \psi_t = h\}. \tag{5}$$

By Lemma 1, if $\sum_{k=1}^{\infty} 1/n_k = \infty$, then all the terms in the right-hand side of (5) are equal to 0; hence we have

$$\mathsf{P}\Big\{\lim_{k\to\infty} \psi_k > 1\Big\} = 0.$$

The proof of Theorem 1 is complete.

3. LIMIT DISTRIBUTION OF THE TIME OF UNIFICATION OF ALL THE PARTICLES

The proof of Theorem 2 is divided into several lemmas. In Lemmas 2 and 3, we obtain rough estimates, which imply that, for any $\varepsilon > 0$, the probability of the time of transition from n particles to M particles being larger than εn can be made arbitrarily small for all sufficiently large n by choosing a sufficiently large $M = \text{const}$. Then we show that the time of residence of the sequence $\{\psi_n\}$ at any level $k = \text{const}$, divided by n, converges in distribution to ξ_k and that the distributions of the sums of these normed times converge to the distributions of the sums of the corresponding independent random variables ξ_k.

We set

$$T(0) = 0, \qquad T(j) = \min(t : \psi_t \leq j), \quad \eta_j = T(j) - T(j+1), \quad j = n-1, \dots, 1.$$

Here η_j is the "time of transition" from $j+1$ unified particles to j unified particles, and

$$\{\eta_j > 0\} = \{\min(t : \psi_t \leq j) > \min(t : \psi_t \leq j+1)\} = \{\psi_{T(j+1)} = j+1\}.$$

Obviously, $\tau_n = T(1) = \sum_{j=1}^{n-1} \eta_j$.

For the case in which each layer consists of exactly n cells, formula (1) takes the form

$$\mathsf{P}\{\psi_{k+1} = m \mid \psi_k = m\} = \prod_{j=1}^{m-1}\left(1 - \frac{j}{n}\right) \stackrel{\text{def}}{=} a_m(n) = a_m \qquad \text{if} \quad 1 \leq m \leq n; \tag{6}$$

therefore,

$$\mathsf{P}\{\eta_k > t \mid \eta_k > 0\} = \prod_{j=1}^{t} \mathsf{P}\{\eta_k > j \mid \eta_k > j-1\} = (a_{k+1})^t, \qquad t = 1, 2, \dots, \tag{7}$$

$$\mathsf{E}\eta_k = \sum_{k=0}^{\infty} \mathsf{P}\{\eta_k > t\} = \sum_{k=0}^{\infty} \mathsf{P}\{\eta_k > 0\}(a_{k+1})^t = \frac{\mathsf{P}\{\eta_k > 0\}}{1 - a_{k+1}}. \tag{8}$$

Lemma 2. *Let $m_i^{(n)}$ be the integer part of the number n/i^2, $i = 1, 2, \dots$. Then*

$$\sum_{1 \le i < n^{1/6}} \mathsf{P}\{T(m_{i+1}^{(n)}) - T(m_i^{(n)}) > 1\} \le \frac{1}{n^{1/6}}.$$

Proof. We let $\mu_0(M, n)$ denote the number of empty cells in the case of allocation of M particles into n cells in the equiprobable one-stage scheme of particle allocation. The event $\{T(m_{i+1}^{(n)}) - T(m_i^{(n)}) > 1\}$ occurs if and only if $\psi_{T(m_i^{(n)})+1} > m_{i+1}^{(n)}$, i.e., in the first equiprobable allocation of $\psi_{T(m_i^{(n)})} \le m_i^{(n)}$ particles into n cells, the number $\mu_0(\psi_{T(m_i^{(n)})}, n)$ of empty cells does not exceed $n - m_{i+1}^{(n)} + 1$. Since the number of empty cells does not increase as the number of allocated particles increases, we have

$$\begin{aligned}\mathsf{P}\{T(m_{i+1}^{(n)}) - T(m_i^{(n)}) > 1\} &= \mathsf{P}\{\mu_0(\psi_{T(m_i^{(n)})}, n) \le n - m_{i+1}^{(n)} + 1,\ \psi_{T(m_i^{(n)})} > m_{i+1}^{(n)}\} \\ &\le \mathsf{P}\{\mu_0(m_i^{(n)}, n) \le n - m_{i+1}^{(n)} + 1\}. \qquad (9)\end{aligned}$$

Now we use the Chebyshev inequality to estimate the probability in the right-hand side:

$$\begin{aligned}\mathsf{P}\{\mu_0(m_i^{(n)}, n) \le n - m_{i+1}^{(n)} + 1\} &= \mathsf{P}\big\{\mathsf{E}\mu_0(m_i^{(n)}, n) - \mu_0(m_i^{(n)}, n) \ge \mathsf{E}\mu_0(m_i^{(n)}, n) - n + m_{i+1}^{(n)} - 1\big\} \\ &\le \frac{\mathsf{D}\mu_0(m_i^{(n)}, n)}{(\mathsf{E}\mu_0(m_i^{(n)}, n) - n + m_{i+1}^{(n)} - 1)^2}. \qquad (10)\end{aligned}$$

To estimate the numerator, we use the exact formulas from [6, pp. 12–13] and the inequality $(1 - 2x) \le (1 - x)^2$:

$$\begin{aligned}\mathsf{D}\mu_0(M, n) &= n(n-1)\Big(1 - \frac{2}{n}\Big)^M + n\Big(1 - \frac{1}{n}\Big)^M - n^2\Big(1 - \frac{1}{n}\Big)^{2M} \\ &\le n\Big(1 - \frac{1}{n}\Big)^M - n\Big(1 - \frac{2}{n}\Big)^M = \sum_{k=0}^{M-1}\Big(1 - \frac{1}{n}\Big)^k\Big(1 - \frac{2}{n}\Big)^{M-k-1} \le M.\end{aligned}$$

Further, the inequalities

$$\Big(1 - \frac{1}{n}\Big)^m \le e^{-m/n} \le 1 - \frac{m}{n} + \frac{m^2}{2n^2}, \quad m, n \ge 0, \qquad \frac{n}{i^2} - 1 < m_i^{(n)} \le \frac{n}{i^2},$$

$$\frac{1}{i^2} - \frac{1}{2i^4} = \frac{2i^2 - 1}{2i^4} > \frac{1}{i(i+1)} = \frac{1}{(i+1)^2} + \frac{1}{i(i+1)^2}$$

imply that

$$\begin{aligned}\mathsf{E}\mu_0(m_i^{(n)}, n) = n\Big(1 - \frac{1}{n}\Big)^{m_i^{(n)}} &\le n - m_i^{(n)} + \frac{(m_i^{(n)})^2}{2n} \le n - \frac{n}{i^2} + \frac{n}{2i^4} + 1 \\ &\le n - \frac{n}{(i+1)^2} - \frac{n}{i(i+1)^2} + 1 < n - m_{i+1}^{(n)} + 1 - \frac{n}{i(i+1)^2}.\end{aligned}$$

Hence we can rewrite inequality (10) as

$$\mathsf{P}\{\mu_0(m_i^{(n)}, n) \le n - m_{i+1}^{(n)} + 1\} \le \frac{n/i^2}{(n/i(i+1)^2)^2} = \frac{(i+1)^4}{n}.$$

This and (9) imply that

$$\sum_{1 \le k < n^{1/6}} \mathsf{P}\{T(m_{k+1}^{(n)}) - T(m_k^{(n)}) > 1\} \le \sum_{1 \le k < n^{1/6}} \frac{(i+1)^4}{n} < \frac{1}{n^{1/6}}.$$

The proof of Lemma 2 is complete. □

By Lemma 2, we have $\mathsf{P}\{T(m^{(n)}_{1+[n^{1/6}]}) - T(n) > n^{1/6}\} \to 0$, i.e., the time of transition from n original particles to $m^{(n)}_{1+[n^{1/6}]} \le n^{2/3}$ unified particles does not exceed $n^{1/6}$ with a probability tending to 1 as $n \to \infty$.

Lemma 3. *For any fixed M, the inequality*

$$\frac{1}{n}\,\mathsf{E}(T(M) - T(m^{(n)}_{1+[n^{1/6}]})) \le \frac{1}{n}\sum_{s=M}^{[n^{2/3}]} \mathsf{E}\eta_s \le \frac{5}{M}$$

holds for all sufficiently large n.

Proof. It follows from (8) that

$$\frac{1}{n}\sum_{s=M}^{[n^{2/3}]} \mathsf{E}\eta_s < \frac{1}{n}\sum_{s=M}^{[n^{2/3}]} \frac{1}{1-a_{s+1}}\,.$$

Since

$$\ln a_{s+1} = \sum_{i=1}^{s} \ln\biggl(1 - \frac{i}{n}\biggr) \le -\frac{1}{2n}s(s+1) \qquad \text{and} \qquad e^{-x} \le \max\{1/2, 1 - x/2\}, \qquad x \ge 0,$$

it follows that

$$1 - a_{s+1} \ge 1 - e^{-s(s+1)/2n} \ge \min\biggl\{\frac{1}{2}\,, \frac{s(s+1)}{4n}\biggr\},$$

and hence

$$\frac{1}{n}\sum_{s=M}^{[n^{2/3}]} \mathsf{E}\eta_s < \sum_{s=M}^{[n^{2/3}]} \max\biggl\{\frac{2}{n}\,, \frac{4}{s(s+1)}\biggr\} < \frac{2}{n^{1/3}} + \frac{4}{M}\,.$$

This estimate implies the assertion of Lemma 3. □

By Lemmas 2 and 3, for any $\varepsilon > 0$ and for sufficiently large n, we have

$$\begin{aligned} \mathsf{P}\biggl\{\frac{1}{n}\sum_{k=M}^{n-1} \eta_k > \varepsilon\biggr\} &= \mathsf{P}\{T(M) - T(n) > \varepsilon n\} \\ &< \mathsf{P}\{T(m^{(n)}_{1+[n^{1/6}]}) - T(n) > n^{1/6}\} + \mathsf{P}\{T(M) - T(m^{(n)}_{1+[n^{1/6}]}) > \varepsilon n\} \\ &< \frac{1}{n^{1/6}} + \frac{5}{M\varepsilon}\,, \end{aligned}$$

i.e., by choosing M, the probability of the event that the transition from n particles to M particles exceeds εn can be made arbitrarily small for all sufficiently large n.

Lemma 4. *If $k < n$, then*

$$\mathsf{P}\{\eta_k = 0\} \le \frac{k^2}{3(n-k)}\,.$$

Proof. According to the remark after the definition of η_s, we have

$$\{\eta_k = 0\} = \{T(k) = T(k+1)\} = \bigcup_{m>k+1,t>0} \{\psi_{t-1} = m,\, T(k) = t,\, T(k+1) = t\},$$

and the events in the right-hand side do not intersect pairwise. Hence

$$\mathsf{P}\{\eta_k = 0\} = \sum_{m>k+1}\sum_{t>0} \mathsf{P}\{\psi_{t-1} = m,\, T(k+1) = t\}\,\mathsf{P}\{T(k) = t \mid \psi_{t-1} = m,\, T(k+1) = t\}. \quad (11)$$

The sum of the first multipliers over m and t is equal to 1; therefore, it suffices to estimate the conditional probabilities

$$\begin{aligned}\mathsf{P}\{T(k) = t \mid \psi_{t-1} = m,\, T(k+1) = t\} &= \mathsf{P}\{\mu_0(m,n) \ge n-k \mid \mu_0(m,n) \ge n-k-1\}\\ &= \mathsf{P}\{\overline{\mu}_1(m,n) \le k \mid \overline{\mu}_1(m,n) \le k+1\},\end{aligned} \tag{12}$$

where $\overline{\mu}_1(m,n)$ is the number of occupied cells in the case of equiprobable allocation of m particles into n cells.

Lemma 5. *If $k < m < n$, then*

$$\frac{\mathsf{P}\{\overline{\mu}_1(m,n) \le k\}}{\mathsf{P}\{\overline{\mu}_1(m,n) \le k+1\}} \le \frac{\mathsf{P}\{\overline{\mu}_1(m,n) = k\}}{\mathsf{P}\{\overline{\mu}_1(m,n) = k+1\}} \le \frac{k^2}{3(n-k)}\,.$$

Proof. The set $A_{m,n}$ of all possible allocations of m particles into n cells consists of $|A_{m,n}| = n^m$ elements of the form $(x_1, \dots, x_m)$, where x_j is the number of the cell into which the jth particle is put. In the case of independent equiprobable allocations, each of these allocations has the probability n^{-m}. By $A_{m,n}(k)$ we denote the set of allocations for which the number of occupied cells is equal to k; then

$$\mathsf{P}\{\overline{\mu}_1(m,n) = k\} = n^{-m}|A_{m,n}(k)|.$$

We also let $A^*_{m,n}(k+1)$ denote the set of allocations for which the number of occupied cells is equal to $k+1$ and, in addition, there exists a cell containing exactly one particle; obviously,

$$A^*_{m,n}(k+1) \subseteq A_{m,n}(k+1). \tag{13}$$

We consider a two-partite graph with the set of vertices $A_{m,n}(k) \cup A^*_{m,n}(k+1)$; in this graph, the vertices $(x_1, \dots, x_m) \in A_{m,n}(k)$ and $(x^*_1, \dots, x^*_m) \in A^*_{m,n}(k+1)$ are connected by an edge if and only if there exits a $j \in \{1, \dots, m\}$ such that $x_j \ne x^*_j$ and $x_i = x^*_i$ for all $i \ne j$, and the set $\bigcup_{i \ne j}\{x_i\}$ contains x_j but does not contain x^*_j.

Let us obtain two-sided estimates for the number $N_{m,n}(k)$ of edges in this graph. Since, for $m > k$ and for for any allocation, $(x_1, \dots, x_m) \in A_{m,n}(k)$, the number of cells with exactly one particle cannot exceed $k-1$, the number of the indices j such that $x_j \in \bigcup_{i \ne j}\{x_i\}$ is no less than $m-k+1$, and the number of empty cells is equal to $n-k$. Hence no less than $(n-k)(m-k+1)$ edges issue from each of the vertices $(x_1, \dots, x_m) \in A_{m,n}(k)$, i.e., $N_{m,n}(k)$ is no less than $(n-k)(m-k+1)|A_{m,n}(k)|$. On the other hand, from any set $(x^*_1, \dots, x^*_m) \in A^*_{m,n}(k+1)$, one can select a set belonging to $A_{m,n}(k)$ by choosing $j \in \{1, \dots, m\}$ such that $x^*_j \notin \bigcup_{i \ne j}\{x^*_i\}$ and replacing x^*_j by any of the k elements of the set $\bigcup_{i \ne j}\{x^*_i\}$. The number of versions of such transformations cannot exceed k^2, because

$$\Bigl|\Bigl\{j : x^*_j \notin \bigcup_{i \ne j}\{x^*_i\}\Bigr\}\Bigr| \le k.$$

Hence at most k^2 edges issue from each of the vertices $(x_1, \dots, x_m) \in A^*_{m,n}(k+1)$, i.e., $N_{m,n}(k)$ does not exceed $k^2|A^*_{m,n}(k+1)|$. It follows from the inequality

$$(n-k)(m-k+1)|A_{m,n}(k)| \le N_{m,n}(k) \le k^2|A^*_{m,n}(k+1)|$$

and inclusion (13) that

$$\frac{\mathsf{P}\{\overline{\mu}_1(m,n) = k\}}{\mathsf{P}\{\overline{\mu}_1(m,n) = k+1\}} \le \frac{k^2}{(n-k)(m-k+1)} \le \frac{k^2}{3(n-k)}$$

for $m \ge k+2$. The right-hand side of this inequality increases monotonically in k, and hence

$$\frac{\mathsf{P}\{\overline{\mu}_1(m,n) \le k\}}{\mathsf{P}\{\overline{\mu}_1(m,n) \le k+1\}} \le \frac{\mathsf{P}\{\overline{\mu}_1(m,n) = 1\} + \dots + \mathsf{P}\{\overline{\mu}_1(m,n) = k\}}{\mathsf{P}\{\overline{\mu}_1(m,n) = 2\} + \dots + \mathsf{P}\{\overline{\mu}_1(m,n) = k+1\}} \le \frac{k^2}{3(n-k)}\,.$$

The proof of Lemma 5 is complete. □

Relations (11) and (12) and Lemma 5 imply the assertion of Lemma 4. □

Lemma 6. *For any fixed k, the random variable η_k/n converges in distribution as $n \to \infty$ to the random variable ξ_k.*

Proof. It follows from (7) that $\mathsf{P}\{\eta_k \le nx\} = 1 - \mathsf{P}\{\eta_k > 0\} a_{k+1}^{[nx]}$ for $x > 0$. By (6), as $n \to \infty$,

$$a_{k+1}^{[nx]} = \left(\prod_{j=1}^{k}\left(1 - \frac{j}{n}\right)\right)^{[nx]} = \exp\left(-[nx]\frac{k(k+1)}{2n}(1 + o(1))\right) \to e^{-k(k+1)x/2}$$

for $k = \text{const}$, and $\mathsf{P}\{\eta_k > 0\} \to 1$ as $n \to \infty$ by Lemma 4. Hence, for $x > 0$, as $n \to \infty$,

$$\mathsf{P}\left\{\frac{1}{n}\eta_k \le x\right\} = \mathsf{P}\{\eta_k \le nx\} \to 1 - e^{-k(k+1)x/2} = \mathsf{P}\{\xi_k \le x\}.$$

The proof of Lemma 6 is complete. □

Lemma 7. *The random variables $(1/n)\sum_{s=1}^{M} \eta_s$ converge in distribution as $n \to \infty$ to the sum $\sum_{s=1}^{M} \xi_s$ for any fixed M.*

Proof. The relation $\{\eta_j > 0\} = \{\psi_{T(j+1)} = j + 1\}$ and the fact that $\{\psi_t\}$ is a Markov chain with nonincreasing trajectories imply that, for any j, the random variable η_j is independent of the random variables $\eta_{j+1}, \dots, \eta_n$ under the condition that $\{\eta_j > 0\}$. Hence, for any $k < n$, in the part of the probability space where all of the random variables $\eta_1, \dots, \eta_M$ are positive, the random variables $\eta_1, \dots, \eta_M$ are independent, i.e., for any integer $t_1, \dots, t_M > 0$,

$$\mathsf{P}\{\eta_1 = t_1, \dots, \eta_M = t_M\} = \mathsf{P}\{\eta_1 = t_1\} \cdots \mathsf{P}\{\eta_s = t_s\}. \tag{14}$$

Therefore, for any $x_1, \dots, x_M > 0$, we have

$$\begin{aligned}\mathsf{P}\{1 \le \eta_1 \le nx_1,\, \dots,\, 1 \le \eta_M \le nx_M\} &= \prod_{j=1}^{M} \mathsf{P}\{1 \le \eta_j \le nx_j\} \\ &= \prod_{j=1}^{M} (\mathsf{P}\{\eta_j \le nx_j\} - \mathsf{P}\{\eta_j = 0\}). \end{aligned}\tag{15}$$

The right-hand side of this relation belongs to the interval

$$\left(\prod_{j=1}^{M} \mathsf{P}\{\eta_j \le nx_j\} - \sum_{j=1}^{M} \mathsf{P}\{\eta_j = 0\}, \prod_{j=1}^{M} \mathsf{P}\{\eta_j \le nx_j\}\right). \tag{16}$$

On the other hand, the left-hand side of (15) satisfied the estimates

$$\begin{aligned}\mathsf{P}\{\eta_1 \le nx_1,\, \dots,\, \eta_M \le nx_M\} &\ge \mathsf{P}\{1 \le \eta_1 \le nx_1,\, \dots,\, 1 \le \eta_M \le nx_M\} \\ &= \mathsf{P}\left\{\{\eta_1 \le nx_1,\, \dots,\, \eta_M \le nx_M\} \setminus \left\{\bigcup_{j=1}^{M}\{\eta_j = 0\}\right\}\right\} \\ &\ge \mathsf{P}\{\eta_1 \le nx_1,\, \dots,\, \eta_M \le nx_M\} - \sum_{j=1}^{M} \mathsf{P}\{\eta_j = 0\}. \end{aligned}\tag{17}$$

It follows from (14)–(17) that

$$\left|\mathsf{P}\{\eta_1 \le nx_1,\, \dots,\, \eta_M \le nx_k\} - \prod_{j=1}^{M} \mathsf{P}\{\eta_j \le nx_j\}\right| \le \sum_{j=1}^{M} \mathsf{P}\{\eta_j = 0\}.$$

The right-hand side of this estimate tends to 0 as $n \to \infty$, and hence (with Lemma 6 taken into account) the joint distribution of $\eta_1/n, \dots, \eta_M/n$ converges to the joint distribution of the independent random variables $\xi_1, \dots, \xi_M$. This implies the assertion of Lemma 7. □

Proof of Theorem 2. On the right-hand side of the relation

$$\frac{1}{n}\sum_{k=1}^{n}\eta_k = \frac{1}{n}\sum_{k=1}^{M}\eta_k + \frac{1}{n}\sum_{k=M+1}^{n}\eta_k,$$

the first term tends to $\sum_{k=1}^{M}\xi_k$ as $n \to \infty$ by Lemmas 6 and 7, and the second term satisfies the estimate

$$\mathsf{P}\left\{\frac{1}{n}\sum_{k=M+1}^{n}\eta_k > \varepsilon\right\} < \frac{1}{n^{1/6}} + \frac{5}{M\varepsilon}$$

by Lemmas 2 and 3. Since ε can be chosen arbitrarily small and M can be chosen arbitrarily large, the preceding estimate implies the assertion of Theorem 2. □

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